Assignment 4

1. Show that for every $f \in C[0,\pi]$ satisfying $f(0) = f(\pi) = 0$ and $f'(x)$ exists for all $x \in [0,\pi]$ and $f' \in R[0,\pi]$, the inequality
$$\int_0^\pi |f|^2 \leq \int_0^\pi |f'|^2$$
holds. Can you characterize the case of equality in this inequality?

2. Consider the class of curves
$$\left\{ \gamma \in C^1[0,1] : \gamma(0) = (0,0), \gamma(1) = (b,0), b > 0, \gamma_2(t) > 0, \forall t \in (0,1)\}, \int_0^1 \sqrt{\gamma_1'^2(t) + \gamma_2'^2(t)}dt = \pi \right\}.$$
Show that $A \leq \pi/2$ where A is the area enclosed by the curve γ and the line segment from the origin to $(b,0)$. Can you characterize the optimal case? This "half" isoperimetric problem is called the Dido's problem.

3. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

4. Show that $d(x,y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$.

5. Define d on $\mathbb{Z} \times \mathbb{Z}$ by $d(n,m) = 2^{-d}$, where d is the largest power of 2 dividing $n - m \neq 0$ and set $d(n,n) = 0$. Verify that d defines a metric on $\mathbb{Z}$.

6. Let f be a C^1-function defined on the plane and consider the surface $\Sigma = \{(x,y,f(x,y) : (x,y) \in \mathbb{R}^2\}$. For every two points p and q on Σ, a C^1-piecewise, continuous curve connecting p and q is a continuous function $\gamma : [0,1] \mapsto \Sigma$ such that its three components γ_1, γ_2 and γ_3 are continuous and C^1-piecewise. Use these curves to define a notion of the distance between p and q on Σ and show that it really defines a metric on Σ.

7. For a metric space (X,d), define $m(x,y) = \min\{d(x,y), 1\}$. Show that m is again a metric. Moreover, a sequence converges in d if and only is it converges in m.

8. Show that whenever d is a metric defines on X, then
$$\rho(x,y) \equiv \frac{d(x,y)}{1 + d(x,y)}$$
is also a metric on X. A sequence converges in d if and only if it converges in ρ.

9. Give an example of two inequivalent metrics which have the same concept of convergence. Hint: Work on $\mathbb{R}$ and consider the previous examples.

10. Show that d_2 is stronger than d_1 on $C[a,b]$ but they are not equivalent. Hint: Construct a sequence $\{f_n\}$ in $C[0,1]$ satisfying $\|f_n\|_1 \to 0$ but $\|f_n\|_2 \to \infty$ as $n \to \infty$.

11. Show that a function f from (X,d) to (Y,ρ) which is continuous at x_0 if and only if for each $\varepsilon > 0$, there exists some δ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever x satisfies $d(x,x_0) < \delta$.

12. Consider the functional Φ defined on $C[a,b]$
$$\Phi(f) = \int_a^b \sqrt{1 + f^2(x)}dx.$$
Show that it is continuous in $C[a,b]$ under both the d_1- and d_∞- distances. A real-valued function defined on a space of functions is traditionally called a functional.

13. Consider the functional Ψ defined on $C[-1,1]$ given by $\Psi(f) = f(0)$. Show that it is continuous in the d_∞- but not in the d_1-metric. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(0) = 1,\ \forall n$.

14. Let A be a non-empty set in (X,d) and define

$$d(x,A) \equiv \inf\{d(x,y) :\ y \in A\}.$$

Show that

$$|d(x,A) - d(y,A)| \le d(x,y), \quad x,y \in X,$$

that is, $x \mapsto d(x,A)$ is "Lipschitz continuous" with Lipschitz constant 1 in X.

15. Let A and B be two sets in (X,d) satisfying $d(A,B) > 0$ where

$$d(A,B) \equiv \inf\{d(x,y) :\ (x,y) \in A \times B\}.$$

Show that there exists a continuous function f from X to $[0,1]$ such that $f \equiv 0$ in A and $f \equiv 1$ in B. This problem shows that there are many continuous functions in a metric space.